Volatility, persistence, and survival in financial markets

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We study the temporal fluctuations in time-dependent stock prices (both individual and composite) as a stochastic phenomenon using general techniques and methods of nonequilibrium statistical mechanics. In particular, we analyze stock price fluctuations as a non-Markovian stochastic process using the first-passage statistical concepts of persistence and survival. We report the results of empirical measurements of the normalized qth-order correlation functions $f_q(t)$, survival probability $S(t)$, and persistence probability $P(t)$ for several stock market dynamical sets. We analyze both minute-to-minute and higher-frequency stock market recordings (i.e., with the sampling time δt of the order of days). We find that the fluctuating stock price is multifractal and the choice of δt has no effect on the qualitative multifractal behavior displayed by the $1/q$ dependence of the generalized Hurst exponent H_q associated with the power-law evolution of the correlation function $f_q(t) \sim t^{H_q}$. The probability $S(t)$ of the stock price remaining above the average up to time t is very sensitive to the total measurement time t_m and the sampling time. The probability $P(t)$ of the stock not returning to the initial value within an interval t has a universal power-law behavior $P(t) \sim t^{-\theta}$, with a persistence exponent θ close to 0.5 that agrees with the prediction $\theta = 1 - H_2$. The empirical financial stocks also present an interesting feature found in turbulent fluids, the extended self-similarity.

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I. INTRODUCTION

The financial stocks are complex, nonlinear, open systems characterized by a large number of parameters. Among other features, they present a *multifractal* behavior [1–11]. Whether or not the multifractality is intrinsic or apparent is still an open question [12]. Several interesting multifractal models [8,9] have been developed over the last decade. For example, the multifractal random walk model, introduced by Barcy and collaborators [10,11], has been recently shown to explain, besides the multifractality, other features of financial time series, such as the absence of correlations between price variations and long-range volatility correlations. Other multifractal models have been proposed [13,14] to account for the asymmetry of the volatility-return correlation function. An important step toward a better understanding of such intricate dynamical processes is to search for new methods that are able to provide information about their temporal evolution. One way to explore the temporal evolution of a stochastic system such as a fluctuating stock price, denoted by $x(t)$, is by measuring the *persistence probability* $P(t)$. That is the probability of the stochastic variable $x(t)$ not reaching its original value corresponding to the starting time t_0 up to a later time t_0+t. This concept, closely related to the first-passage probability, has been successfully implemented in surface growth phenomena [15,16] and has been used to determine the universality class and the nonlinear features of the underlying dynamical process through the exponent θ associated with the power-law decay $P(t) \sim t^{-\theta}$ of the persistence probability at large times. Alternatively, one is interested in measuring the survival probability $S(t)$ which is defined as the probability of the stock price remaining above a reference value up to time t. In contrast with the persistence probability, we show that the survival probability depends independently on both the total measurement time t_m and the time between successive recordings—i.e., the sampling time δt. The concepts of persistence and first-passage time have been recently introduced in econophysics [17,18]. Despite the capacity of first-passage statistical tools to predict the degree of performance of a given stock (i.e., how long a stock remains above a certain value, what is the first time when it reaches a particular level), it is quite intriguing that their use in understanding the evolution of financial markets is rather scarce. Our study emphasizes the potential of the survival and persistence probabilities in any time series investigations and shows how these concepts can be connected to the traditional analysis [19] based on the time evolution of price-price correlation functions. We also point out the *extended self-similar* behavior manifested by the stock price correlation functions. Extended self-similarity was originally observed in fluid turbulence problems [20,21] and subsequently in discrete stochastic surface growth models [22]. Therefore we combine features from very different fields, such as surface growth, econophysics of stock markets, and fluid dynamics, in an effort towards understanding the temporal evolution of financial stocks.

Our statistical study of stock market temporal fluctuations is motivated primarily by the availability of the huge amount of quantitative data on the stock market prices both for individual stocks and for aggregate stock indices and both for instantaneous minute-to-minute price fluctuations and for large-scale fluctuations over several years. Such detailed and precise quantitative information about the time dependence of a far-from-equilibrium stochastic process is not easy to find in real physical systems. For example, temporal thermal fluctuations of steps on solid surfaces, which we have recently analyzed [23–25] to study the persistence and survival properties of equilibrium step fluctuations, usually provide reliable time-dependent data only over a couple of decades. Similarly, experimental studies [26] of kinetic surface roughening in interface growth, which have served as model prob-

lems for developing the concepts of dynamical scaling in nonequilibrium growth phenomena, usually have reliable data for a couple of decades in growth time. The availability of extensive quantitative information on stock prices, including minute-to-minute price fluctuations within a single day as well as prices spanning over years (or even decades for some stocks and index averages), therefore provides an almost unique opportunity to statistically analyze a time-dependent non-Markovian stochastic phenomenon using actual "experimental" data covering many decades in time, in contrast to the corresponding experimental studies of real physical systems which span only a few decades in time. It should consequently be possible to obtain definitive information about the persistence and survival behavior of stock price fluctuations studied as a non-Markovian stochastic phenomenon. The availability of extensive stock price data should also allow for a detailed multifractal analysis of stock price fluctuations, which could serve, in principle, as a quantitative measure of "volatility" in the stock market since multifractality is directly related to intermittency and intermittency arises from the finite probability for very large scale fluctuations.

Our work presented in this article should therefore be thought of as a detailed empirical "experimental" analysis of a non-Markovian dynamical stochastic process—namely, the stock market—from the specific perspectives of first-passage statistics (i.e., persistence and survival probabilities and exponents) and intermittency (i.e., multifractality and volatility). Our motivation for studying this specific problem is the existence of a huge amount of experimental data (i.e., stock prices) of a very wide dynamical range. The fact that there is widespread current interest in the statistical physics community in the subject of "econophysics," which is nothing but the study of economics using the principles and methodologies of statistical mechanics, gives our current work some broad context, but our own interest is, however, complementary—we are using the vast amount of available dynamical data on stock price fluctuations to carry out an "experimental" study of the important first-passage statistical concepts of persistence and survival in non-Markovian stochastic phenomena. The current work is, in some sense, a continuation of our earlier work on understanding various stochastic phenomena (i.e., thermal step fluctuations [23–25] and nonequilibrium surface growth [16]) from the first-passage statistics perspective—here the stochastic process under consideration being an economic phenomenon (i.e., stock price fluctuations) rather than physical phenomena as in the past.

Our study of the multifractal character of price fluctuations is based on a multifractal version [27] of the traditional detrended fluctuation analysis [28]. We also use a standard dynamical scaling analysis, inspired from surface growth phenomena, to show the multifractal behavior of the financial stocks, quantitatively extracted from the q-order price-price correlation functions. Our results go in parallel with earlier analyses of other groups [1–7] which succeeded in showing multifractality in several stock markets and commodities. On the other hand, we investigate the persistence of price fluctuations described by the persistence exponent θ. The bridge between these two analyses is provided by the second-order Hurst exponent H_2 associated with the correlation function of the stock price, which has been shown [15,16] to be simply related to the persistence exponent through $H_2=1-\theta$. In this study we verify that this simple relation is satisfied by both low- and high-frequency fluctuating financial stocks.

The data we use for our stochastic study comprise the daily Intel (INTC) stock value between January 1990 and December 2002 and the composite NYSE index (i.e., NYA index) between January 1966 and December 2002. This corresponds to 3355 and 9312 data points, respectively. We also analyze sets of data recorded every minute for the Johnson and Johnson stock (JJ 2000) and every 5 min for the Intel stock (INTC 2000) during the year of 2000 [29]. This corresponds to 98 280 and 19 656 data points, respectively. The daily recorded stocks have an obvious exponential increase of their prices over several years. Therefore the exponential drift of the background can be subtracted from the stock price and we define a new stochastic variable $\tilde{x}(t)=x(t)-x_b(t)$, where the stochastic variable $x_b(t)$ associated with the evolution of the background depends on two parameters a and b, $x_b(t)=a\exp(bt)$ [30]. We analyze the persistence probability of both $x(t)$ and $\tilde{x}(t)$. Changing the background subtraction within reasonable limits does not affect our statistical conclusions about persistence exponents and/or multifractality.

The rest of the paper is organized as follows. In Sec. II, we define the various dynamical correlation functions and related statistical quantities, as well as the various exponents to be used throughout our statistical analyses—Sec. II is important in introducing the methodology of our analyses; in Sec. III, we present our extensive results and discussions; our concluding remarks are exposed in Sec. IV.

II. NONEQUILIBRIUM STATISTICAL MECHANICS TECHNIQUES

A. Price-price correlation functions and extended self-similarity

The generalized qth-order price-price correlation function is defined as

$$G_q(t)=\langle|x(t_0+t)-x(t_0)|^q\rangle^{1/q}, \tag{1}$$

where $x(t)$ is the stock price and the average is over all the initial times t_0. $G_q(t)$ has a power-law behavior

$$G_q(t)\sim t^{H_q}, \tag{2}$$

which defines the exponent hierarchy H_q, also called the generalized Hurst exponent. The price evolution is *multifractal* if the exponent hierarchy H_q varies with q, otherwise is fractal (in the theory of surface dynamical scaling referred to as multiaffine and self-affine, respectively). In particular, for $q=2$, we recover the fractional Brownian motion case described by the well-known Hurst exponent, $0<H_2<1$. A simple way of assigning the presence of multifractality in a stochastic stock market is by looking at the multifractal spectra, $\tau_q=qH_q-1$. For fractals τ_q depend linearly on q. A nonlinear behavior of τ_q vs q is considered a manifestation of multifractality.

The *temporal* behavior in Eq. (2) of the generalized price-price correlation functions is analogous to the *spatial* behavior of the q-order structure functions of turbulent fluids or the generalized height difference correlation functions, $C_q(\mathbf{r},t)=\langle|h(\mathbf{x}+\mathbf{r},t)-h(\mathbf{x},t)|^q\rangle^{1/q}\sim r^{\xi_q}$ (for small distances r), corresponding to surface growth models. The multiscaling behavior is revealed by the q dependence of the scaling exponents ξ_q. The extended self-similarity (ESS) behavior in both turbulent fluids and surface roughening models refers to the enhancement of the scaling region once the $\log|C_m(r)|$ is plotted against $\log|C_n(r)|$, where m and n are two different positive integers [22]. In this study we show that for different financial stocks $G_m(t)$ depends on $G_n(t)$ in a power-law fashion and, although the temporal scaling domain is not necessarily enhanced as in the case of turbulent fluids or roughening models, the ESS behavior is clearly identified and this interesting feature of the financial stocks can be further exploited to understand the associated multifractal character.

B. Persistence exponent of fractional brownian motion

The fractional Brownian motion (FBM) is one of the simplest stochastic models that can be used to model financial stocks or any time series with long-range memory. Before proceeding further, we summarize the definition and the known first-passage property of a FBM. We then describe the multifractal detrended fluctuation analysis (MF-DFA) method which is used to estimate the generalized Hurst exponent.

A stochastic process $\xi(t)$ [with zero mean $\langle\xi(t)\rangle=0$] is called a FBM if its two-time correlation function $C(t_1,t_2)=\langle[\xi(t_1)-\xi(t_2)]^2\rangle$ is (i) *stationary*—i.e., depends only on the time difference $|t_2-t_1|$—and (ii) grows asymptotically as a power law [31]:

$$C(t_1,t_2)\sim|t_2-t_1|^{2H},\quad |t_2-t_1|\gg 1. \tag{3}$$

The parameter $0<H<1$ is called the Hurst exponent, which characterizes the FBM, and $\langle\cdots\rangle$ denotes the expectation value over all realizations of the process $\xi(t)$. In order to match the notation throughout the paper, let us call the Hurst exponent H_2 instead of H. For the sake of completeness we also mention that, alternatively, a zero-mean stochastic process $\xi(t)$ is called a FBM if its autocorrelation function has the following expression:

$$a_\xi(t_1,t_2)=\langle\xi(t_1)\xi(t_2)\rangle\sim t_1^{2H_2}+t_2^{2H_2}-|t_2-t_1|^{2H_2}. \tag{4}$$

The zero-crossing properties of a FBM have been studied extensively in the past [32–34]. In particular, assuming that $\xi(t=0)=0$, we are interested in the probability $P(t)$ that a FBM does *not* cross zero up to time t (i.e., the persistence probability):

$$P(t)=\text{Prob}\{\xi(t')>0,\ \forall\ t'<t\}. \tag{5}$$

In terms of the stochastic stock price variable $x(t)$, characterized by a particular value $x(t_0)$ at the initial time t_0, the probability of remaining always *above* that value up to time t_0+t (i.e., *positive* persistence) reads

$$P_+(t)=\text{Prob}\{x(t_0+t')>x(t_0),\ \forall\ t'<t\}, \tag{6}$$

and, similarly, the *negative* persistence probability reads

$$P_-(t)=\text{Prob}\{x(t_0+t')<x(t_0),\ \forall\ t'<t\}. \tag{7}$$

These definitions can alternatively be reformulated in terms of the *cumulative* time series of the discretized log returns. Let $r_j=\ln[x_{j+\Delta j}/x_j]$ be the discrete set of log returns, with $j=0,1,\ldots$. The sampling time Δj is the interval between successive measurements. We define the cumulative log returns, $R_j=\Sigma_{i=0}^{j}r_i$. Since $R_j=\ln[x_j/x_0]$, the definition of the positive persistence probability, for example, becomes

$$P_+(N)=\text{Prob}\{R_n>0,\ \forall\ 0\leqslant n<N\}. \tag{8}$$

In several studies of *linear* surface growth models, characterized by identical positive and negative persistence probabilities, it has been shown that $P(t)$ decays as a power law [15,16] at large times, $P(t)\sim t^{-\theta}$, with the steady-state persistence exponent θ obeying the relation

$$\theta=1-H_2. \tag{9}$$

We note that this relation holds for any zero-mean process (not necessarily Gaussian [33,35]) that satisfies requirements (i) and (ii) above. Both analytic arguments as well as numerical simulations supporting the relation (9) have been presented previously in the literature in the context of fluctuating interfaces. In this study we investigate the behavior of $P(t)$ at large times and its dependence on the sampling time for both $x(t)$ and $\tilde{x}(t)$ stochastic variables.

The persistence probability can be generalized [23,36] using the persistent large deviations probability $P(t,s)$, defined as the probability for the "average sign" S_{av} of the stock price fluctuation to remain above a certain preassigned value "s" up to time t:

$$P(t,s)\equiv\text{Prob}\{S_{\text{av}}(t')\geqslant s,\ \forall\ t'\leqslant t\}, \tag{10}$$

where

$$S_{\text{av}}(t)\equiv t^{-1}\int_0^t dt'\ \text{sgn}[x(t_0+t')-x(t_0)]. \tag{11}$$

Since $S_{\text{av}}(t)\in[-1,1]$, the probability $P(t,s)$ is defined for $-1\leqslant s\leqslant 1$. For $s=1$ we recover our earlier simple definition of persistence, while for $s=-1$ the probability $P(t,s=-1)$ is trivially equal to unity for all t. However, for the remaining values of the average sign parameter s, $-1<s<1$, the generalization of the persistence probability provides new information through the family of persistent large deviation exponents, θ_l, associated with the power-law behavior, $P(t,s)\sim t^{-\theta_l(s)}$, at large time scales.

C. Survival probability

Perhaps of more practical interest in evaluating the temporal trend of a financial stock is the probability of the stock's price remaining above (below) a certain reference value up to a later time t_0+t, given that its initial value at time t_0 was above (below) that reference level—i.e., the positive (negative) survival probability $S_\pm(t)$. Let us denote by $S(t)$ the average between the positive and negative survival

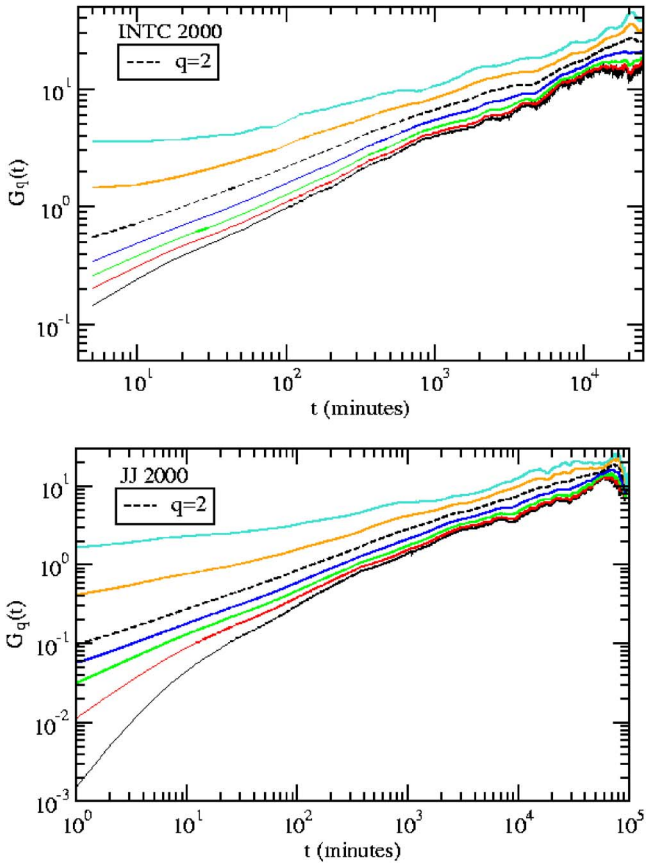

FIG. 1. (Color online) Log-log plot of the generalized price-price correlation function $G_q(t)$ vs t corresponding to minute-to-minute INTC stock (top) and JJ stock (bottom) for $q=1/8,1/4,1/2,1,2,4,8$ from bottom to top in each panel.

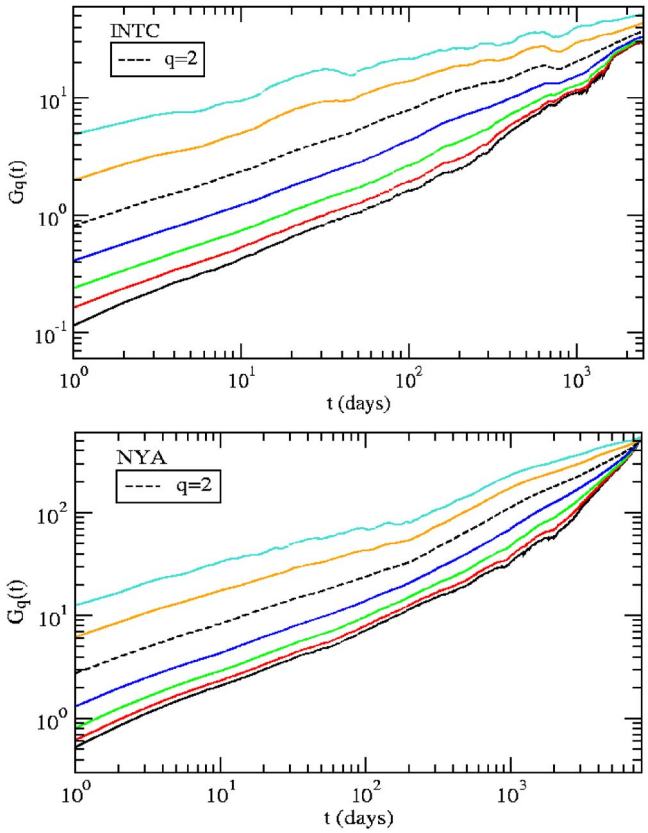

FIG. 2. (Color online) Log-log plot of the generalized price-price correlation function $G_q(t)$ vs t corresponding to the daily INTC stock (top) and NYA index (bottom) for $q=1/8,1/4,1/2,1,2,4,8$ from bottom to top in each panel.

probabilities. This statistical quantity offers a better picture of the likelihood of a given stock having a positive evolving trend, for example, with respect to a preassigned reference value. The definition of this probability reads

$$S_+(t) = \mathrm{Prob}\{x(t_0+t') > \bar{x}, \quad \forall\, t' < t\}, \tag{12}$$

where $\bar{x}$ is the reference price. For simplicity we consider $\bar{x}$ to be the average price over the measurement time t_m, but in general it can take any value between the minimum and the maximum values of $x(t)$ for all the discrete times t up to the final measurement time. We will show that $S(t)$ depends independently on both t_m and δt and the scaling with $t/\delta t$ appears only when $\delta t/t_m$ is a constant. The same type of behavior has been found recently in experimental thermal fluctuations of surface steps on Ag(111), screw dislocations on the facets of Pb crystallites, and Al-terminated Si(111) surfaces [24,25].

D. Multifractal detrended fluctuations analysis

MF-DFA is a reliable method for analyzing correlated time series. It is known to provide the accurate values of the generalized Hurst exponents even for time series with small length, while other similar methods, such as the Hurst rescaled-range analysis [37], overestimate those values in the case of small-size series [38].

Let x_i, $i=1,\ldots,T_f$, be the stochastic price variable recorded at discrete times i. The final transaction time is denoted by T_f. We denote by r_j the log-return price variable, $r_j=\ln(x_{j+1}/x_j)$, $j=1,\ldots,T$, where $T=T_f-1$. We estimate the cumulative time series of the log-return price variables,

$$X(i) = \sum_{j=1}^{i} (r_j - \bar{r}), \quad i = 1,\ldots,T, \tag{13}$$

where $\bar{r}=1/T\Sigma_{i=1}^{T} r_i$ is the average value of the log returns. The time series $X(i)$ is divided into N_τ disjoint segments I_n $(n=1,\ldots,N_\tau)$ of equal size τ. Obviously, $N_\tau=[T/\tau]$. For each segment we calculate the local trend using a linear least-squared fit $Y_\tau(n,t)=a_n+b_n t$, where $t\in I_n$ and $n=1,\ldots,N_\tau$. The local time series of the cumulative log returns is simply $X_\tau(n,t)=X[(n-1)\tau+t]$. Therefore, the variance is given by

$$F^2(n,\tau) = \frac{1}{\tau}\sum_{t=1}^{\tau} [X_\tau(n,t) - Y_\tau(n,t)]^2. \tag{14}$$

$F(n,\tau)$ is called the fluctuation function. In order to avoid disregarding some data points $X(i)$ when the length T of the time series is not a multiple of the time lag τ, one has to repeat these steps starting from the opposite end of the interval. In that case, $X_\tau(n,t)$ in Eq. (14) becomes equal to

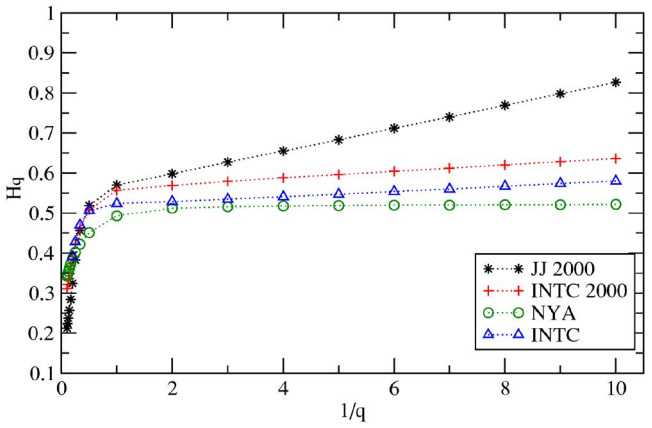

FIG. 3. (Color online) The generalized Hurst exponent H_q vs $1/q$ for the four stocks discussed in the paper. H_q behaves linearly with $1/q$ at both small and large values of $1/q$. For the daily stocks a linear least-squares fit has been applied to the first two decades, while for the high-frequency stocks, the fitted regions were $25<t<500$ for the INTC 2000 stock and $25<t<400$ for the JJ 2000 stock.

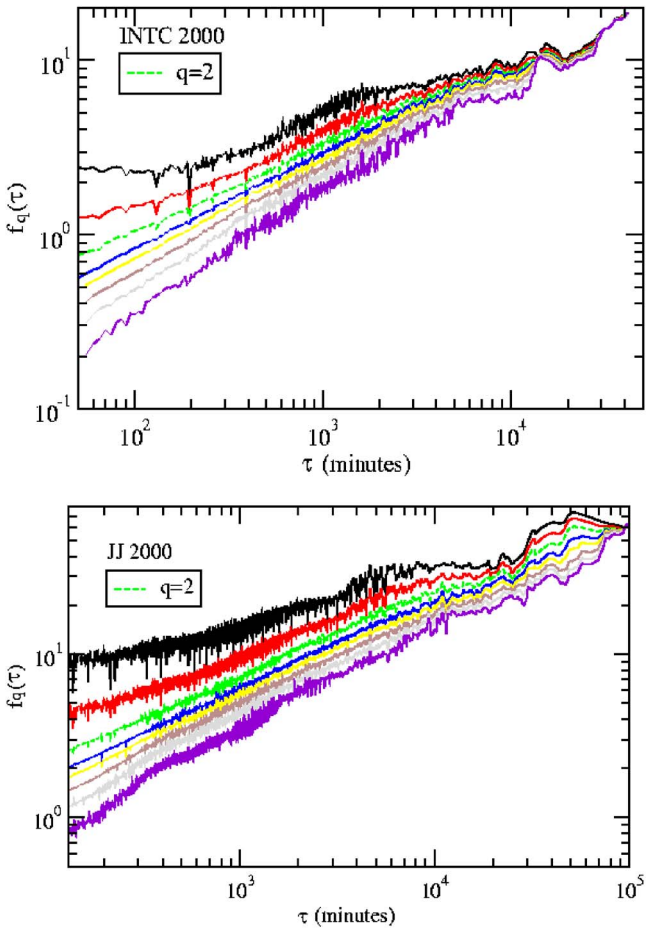

FIG. 4. (Color online) Log-log plot of the normalized fluctuation function $f_q(\tau)$ vs time lag τ corresponding to INTC 2000 stock (top) and JJ 2000 stock (bottom). The MF-DFA method has been used. The curves shown correspond to $q=-8,-4,-2,-1/2,1/2,2,4,8$ from bottom to top in each panel.

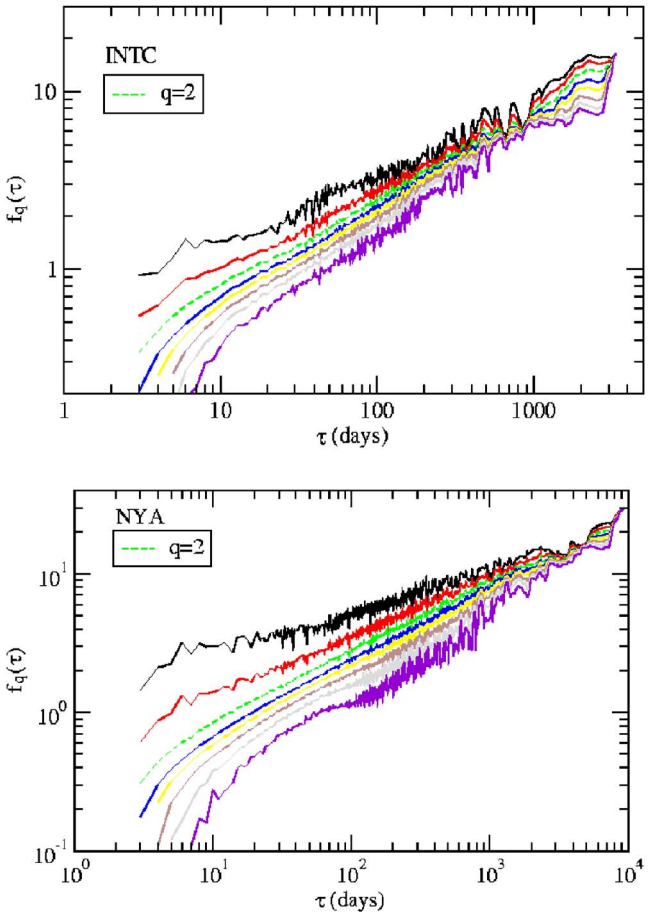

FIG. 5. (Color online) Log-log plot of the normalized fluctuation function $f_q(\tau)$ vs τ corresponding to the daily INTC stock (top) and NYA index (bottom). The MF-DFA method has been used to calculate $f_q(\tau)$. The curves shown correspond to $-8,-4,-2,-1/2,1/2,2,4,8$ from bottom to top in each panel.

$X[T-(n-N_\tau)\tau+t]$, for $n=N_\tau+1,\ldots,2N_\tau$. By averaging over all the segments I_n we finally obtain the correlation function of order q,

$$F_q(\tau)=\left\{\frac{1}{2N_\tau}\sum_{n=1}^{2N_\tau}[F^2(n,\tau)]^{q/2}\right\}^{1/q}. \tag{15}$$

By construction, since we use a linear fit for simplicity, $F_q(\tau)$ is defined for $\tau\geqslant 3$. The scaling form of the correlation function $F_q(\tau)\sim\tau^{H_q}$ provides the family of generalized Hurst exponents H_q. For reasons that will become clearer very shortly we also introduce the dimensionless fluctuation function $f(n,\tau)$, defined by

$$f(n,\tau)=\frac{[F^2(n,\tau)]^{1/2}}{\sigma}, \tag{16}$$

where $\sigma=\sqrt{1/T\Sigma_{t=1}^{T}(r_t-\bar{r})^2}$ is the standard deviation of the log returns during the interval T. Therefore, the dimensionless qth-order correlation function becomes

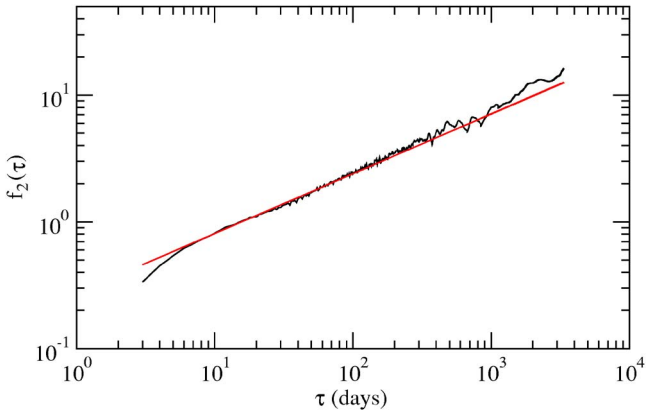

FIG. 6. (Color online) The normalized fluctuation function $f_2(\tau)$ as a function of time for the daily INTC stock during the period 1990–2002. The straight line represents the theoretical curve based on Eqs. (18) and (19) for $H_2=0.47$.

$$f_q(\tau) = \left\{ \frac{1}{2N_\tau} \sum_{n=1}^{2N_\tau} [f(n,\tau)]^q \right\}^{1/q}. \tag{17}$$

Obviously, f_q obeys the same scaling relation as F_q,

$$f_q(\tau) = C_{H_q} \tau^{H_q}, \tag{18}$$

where C_{H_q} is a constant independent of the time lag τ. However, for $q=2$ (which corresponds to the usual DFA procedure), the expression of this constant is known exactly [39],

$$C_{H_2} = \left(\frac{2}{2H_2+1} + \frac{1}{H_2+2} - \frac{2}{H_2+1} \right)^{1/2}, \tag{19}$$

where H_2 is the Hurst exponent of the FBM. The time evolution of $f_q(\tau)$, along with the analytical result for the coefficient C_{H_2}, can be used to understand the dynamics and memory of financial stocks.

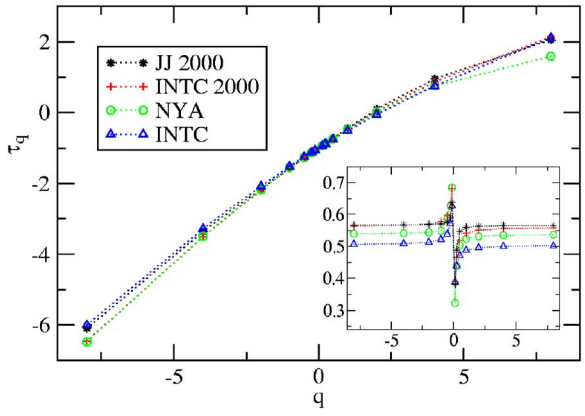

FIG. 7. (Color online) The multifractal spectra τ_q vs q for the stocks discussed in the paper. The deviation of all curves from the linear dependence is a signature of the multifractal behavior. The inset shows the $1/q$ dependence of H_q, keeping the same symbol-stock correspondence as in the main figure.

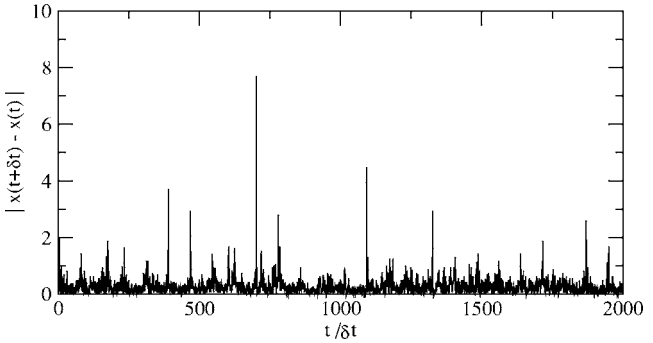

FIG. 8. The local stock price difference $|x(t+\delta t)-x(t)|$ vs the dimensionless variable $t/\delta t$ for the INTC 2000 recorded with sampling time $\delta t=5$ min.

III. RESULTS AND DISCUSSIONS

We first discuss the results concerning the price-price correlation functions calculated using Eq. (1). In Fig. 1 we present the results of $G_q(t)$ for the high-frequency stocks and in Fig. 2 for the low-frequency stocks. It is obvious that since the log-log plots of $G_q(t)$ vs t do not exhibit linear behavior over the entire time range, the associated Hurst exponent H_q varies with time and the scaling of the correlation functions suffers many transient regimes. A good power-law dependence appears for $q=2$. Although for other values of q the deviations from a power law become visible, it is clear that H_q decreases with q. For illustration purposes we have fitted certain portions of these log-log plots to obtain a qualitative view of the dependence of H_q on $1/q$, as shown in Fig. 3. In this figure we have used a large range of values for q (i.e., $q=1/10, 1/9, \ldots, 1, 2, \ldots, 10$). We notice that H_q depends linearly on $1/q$ for both small and large values of $1/q$. The stock with the smallest sampling time of $\delta t=1$ min (JJ 2000) displays an increase of H_q at large values of $1/q$, while for the rest of the stocks H_q has the tendency to saturate as $1/q$ increases.

When using the MF-DFA method to calculate the correlation functions of order q we note that the power-law behavior of $f_q(\tau)$ vs τ extends over longer time periods, making the extraction of the exponent H_q more reliable. The results are

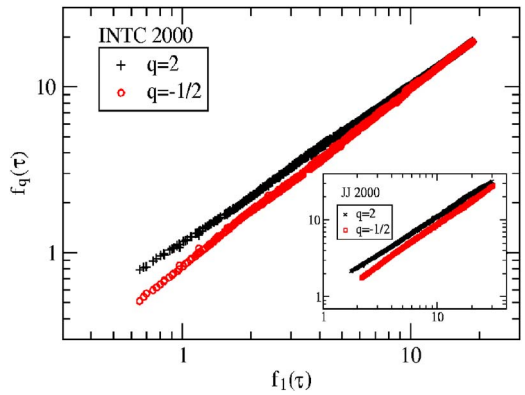

FIG. 9. (Color online) The normalized fluctuation functions $f_2(\tau)$ and $f_{-1/2}(\tau)$ as a function of $f_1(\tau)$ for the INTC 2000 and JJ 2000 (inset) stocks.

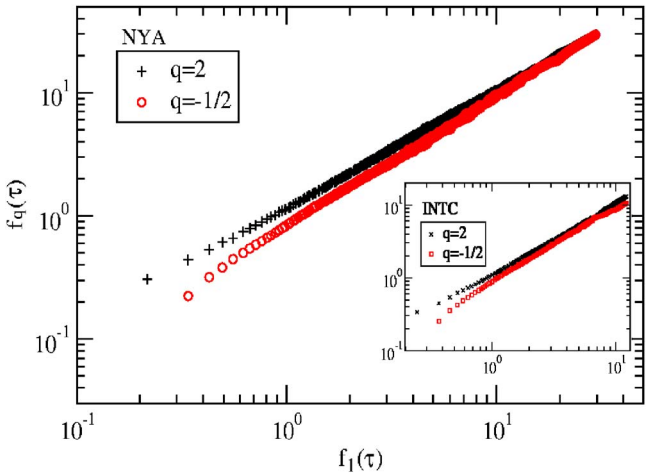

FIG. 10. (Color online) The normalized fluctuation functions $f_2(\tau)$ and $f_{-1/2}(\tau)$ as a function of $f_1(\tau)$ for the daily NYA and INTC (inset) stocks.

shown in Figs. 4 and 5. This can be easily seen in the case of the daily NYA index and INTC stocks. The power law is seen for more than three decades in Fig. 5, while limited power laws spanning two decades only are seen in Fig. 2.

As we have already mentioned, the value of H_q is sensitive to the fitted region of the log-log plot of $f_q(\tau)$ vs τ. This issue requires special treatment. Our strategy was to take advantage of the analytic result of Eq. (19) in order to identify the time window over which the second-order Hurst exponent can be extracted correctly. This can be achieved by adjusting H_2 such that the agreement between the empirical results for $f_2(\tau)$ using Eq. (17) and the theoretical curve predicted by Eq. (18) with $q=2$ and Eq. (19) becomes very good. This procedure is shown in Fig. 6. Once the time window which gives the best agreement for $q=2$ is identified, we use it to extract H_q for a large set of q values $(q=\pm 8, \pm 4, \pm 2, \pm 1, \pm 1/2, \pm 1/4, \pm 1/8)$. The results for the q dependence of H_q based on the MF-DFA method have been used to calculate the multifractal spectra, $\tau_q=qH_q-1$, shown in Fig. 7. We observe that all spectra deviate from a linear q dependence, which is an obvious manifestation of the multifractality in these stocks. In the inset of Fig. 7 we also show the $1/q$ dependence of H_q. At positive q, the qualitative trend of the results is the same as in Fig. 3. It is interesting to point

TABLE I. Extended self-similar behavior of financial stocks based on the power-law dependence of $\log|f_m(\tau)|$ on $\log|f_n(\tau)|$ ($m=-\frac{1}{2}$ and 2 and $n=1$). The exponents $\alpha_{-1/2,1}$ and $\alpha_{2,1}$ extracted from the power laws shown in Figs. 9 and 10 are in very good agreement with the expected ratios of $H_{-1/2}/H_1$ and H_2/H_1, respectively.

Stock	H_2/H_1	$H_{-1/2}/H_1$	$\alpha_{2,1}$	$\alpha_{-1/2,1}$
INTC	0.966	1.051	0.956	1.065
NYA	0.960	1.059	0.961	1.058
INTC 2000	0.958	1.062	0.956	1.065
JJ 2000	0.983	1.033	0.962	1.040

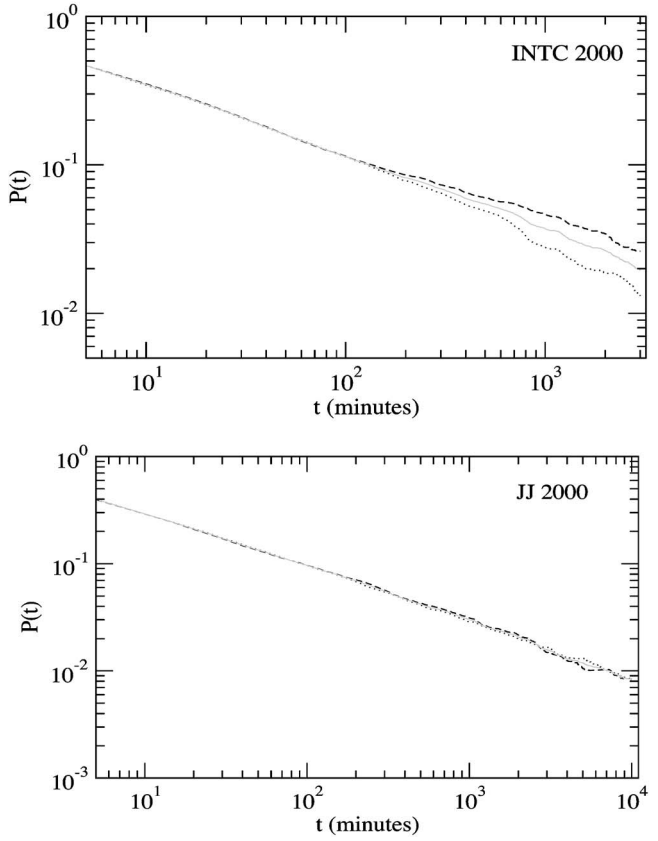

FIG. 11. Persistence probabilities $P(t)$ vs t for the minute-to-minute stocks. The dashed line corresponds to $P_+(t)$, the dotted line corresponds to $P_-(t)$, and the solid line represents $P(t)$.

out that the empirical set with the smallest sampling time, JJ 2000, which in the case of the standard price-price correlation function analysis has shown an increasing trend of H_q at large values of $1/q$, does not present this trend anymore, H_q saturating quickly as $1/q$ increases. Negative values of q are accessible within the MF-DFA analysis. For large negative values of $1/q$ the generalized Hurst exponents saturate rather fast, as in the case of large positive values of $1/q$.

The fact that H_q (and in particular H_2) changes with time is a clear indication of the multifractal character. In this context we mention that the so-called multifractional Brownian motion could be alternatively used to model this feature [38].

The analogy of the stock market fluctuations and fluid turbulence has already been pointed out in the literature [40]. It is known that in turbulent fluids the energy dissipation rate shows violent fluctuations. Similarly, as shown in Fig. 8, we find that the temporal evolution of the local stock price differences, $|x(t+\delta t)-x(t)|$, presents strong fluctuations which represent the signature of intermittency. In addition, the self-extending similarity features of intermittent fluid turbulence have been shown to exist in *spatial* height correlation functions of the kinetic surface roughening models. We show in Figs. 8 and 9 that the extended self-similarity exhibited by the structure functions in fluid turbulence also shows up in the *temporal* behavior of the financial stocks correlation functions. This observation offers a connection between three distinct physical problems, apparently without any in-

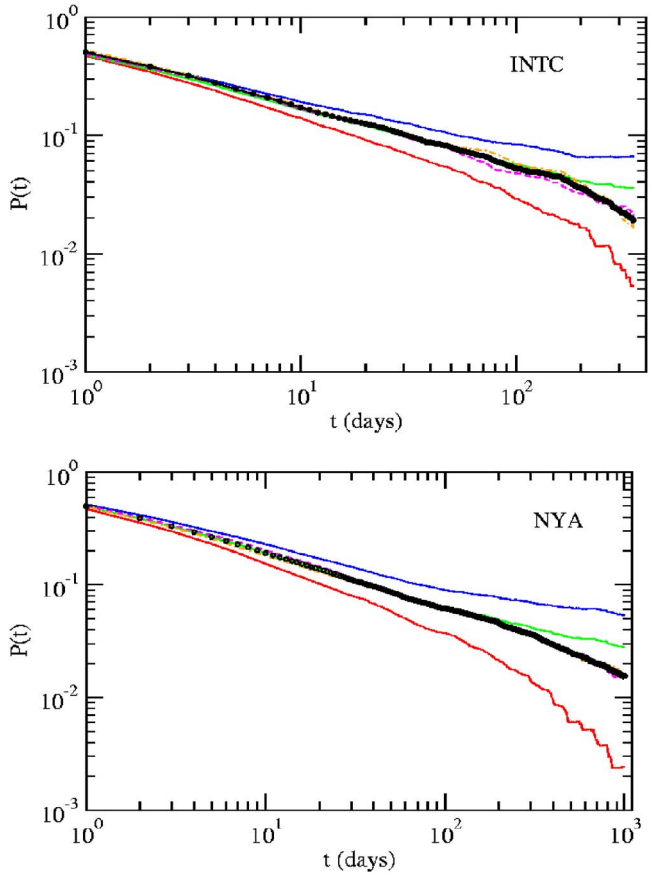

FIG. 12. (Color online) Persistence probabilities $P(t)$ vs t for the daily stocks. Solid lines correspond to $P_+(t)$, $P(t)$, and $P_-(t)$ (from top to bottom). The dashed lines correspond to $\tilde{P}_+(t)$ and $\tilde{P}_-(t)$. The circle represents the average probability $\tilde{P}(t)$ for the variable $\tilde{x}(t)$.

tuitive connections: fluid turbulence, surface roughening, and financial stocks.

For exemplification purpose we plot, in Figs. 9 and 10, $\log|f_m(\tau)|$ vs $\log|f_n(\tau)|$ for $m=-1/2$ and 2 and $n=1$. From the linear behavior of these plots it is obvious that $f_m(\tau)\sim[f_n(\tau)]^{\alpha_{mn}}$, where the expectation value for the exponents α_{mn} is $\alpha_{mn}=H_m/H_n$. We find that our empirical results for α_{mn} are in very good agreement with the expected ratios between H_m and H_n, as summarized in Table I. It would be interesting to check the existence of the extended self-similarity in other financial stocks. We mention that the analysis of correlation functions based on the extended self-similarity is known to provide reliable values of the ratios between several generalized Hurst exponents [22].

Next, we present in the results of the persistence probabilities and persistence exponents for our fluctuating stocks. In Fig. 11 we show the results based on the minute-to-minute stocks and in Fig. 12 the daily stocks. We observe that the best power law appears for the average persistence probability $P(t)=1/2[P_+(t)+P_-(t)]$, while departures from the power-law behavior can be seen for $P_\pm(t)$ corresponding to the INTC 2000 stock and more clearly for the daily stocks. For the low-frequency stocks, in addition to measuring the positive, negative, and average persistence probabilities of the stochastic price variables, we have

TABLE II. The persistence exponent θ associated with the power-law decay of the average probability $P(t)$ $[\tilde{P}(t)]$ for the minute-to-minute stocks [daily stocks]. H_2 is the second-order Hurst exponent extracted from the time evolution of $f_2(\tau)$ [see Eq. (18)].

Stock	θ	H_2
INTC	0.51	0.47
NYA	0.50	0.50
INTC 2000	0.50	0.47
JJ 2000	0.52	0.47

also considered the set of these three probabilities corresponding to the empirical sets after the background elimination—i.e., $\tilde{x}(t)=x(t)-a\exp(bt)$. For the INTC stock we have that $a=1.1703$ and $b=1.1889\times10^{-3}$, and for NYA index $a=28.473$ and $b=0.3177\times10^{-3}$. The persistence curves for the variable $\tilde{x}(t)$ are very similar in the sense that no distinction between the positive, negative, and average probabilities can be made. This result agrees with previous studies of the persistence probability of the German stock index [17]. We have used $P(t)$ and $\tilde{P}(t)$ in order to extract the persistence exponents for the minute-to-minute and daily stocks, respectively. The results are summarized in Table II. We compared these values against $1-H_2$, with H_2 extracted from the fitted power law of $f_2(\tau)$, in order to investigate the validity of Eq. (9). We find good agreement between θ and $1-H_2$. However, it is important to emphasize that since both θ and H_2 are very close to $1/2$, the memory effects of the time series under investigation can only be revealed by higher-order correlation functions. The second-order correlation function by itself cannot explain the multifractality discussed in this study since it indicates that the returns are uncorrelated. We also add that $P(t)$ is not sensitive to the large discrepancy between δt and t_m corresponding to the high-frequency stocks and daily stocks, respectively.

The generalization of the persistence probability is shown in Fig. 13. We only present the results for the probability of persistent large deviations corresponding to the INTC 2000 stock, but we have checked the applicability of this concept to other empirical stocks as well. From the linear behavior of $\log P(t,s)$ vs $\log t$ we conclude that the t dependence of $P(t,s)$ is indeed a power law. We see that the local slope decreases as the average sign parameter s decreases. We have varied s from -1 to 1 with an increment of 0.1, and the s dependence of the resulting family of persistent large deviations exponents is shown in the inset of Fig. 13. We mention that each curve in Fig. 13 corresponds to the average between the positive and negative persistent large deviations probabilities—i.e., $P(t,s)=1/2[P_+(t,s)+P_-(t,s)]$. Both $P_+(t,s)$ and $P_-(t,s)$ show departures from the expected power laws at large t, as we have seen in Figs. 11 and 12 in the case of the positive and negative persistence probabilities.

Finally we show the results of the survival probability in Fig. 14. We have looked at the temporal evolution of the JJ 2000 stock price recorded with two different sampling times,

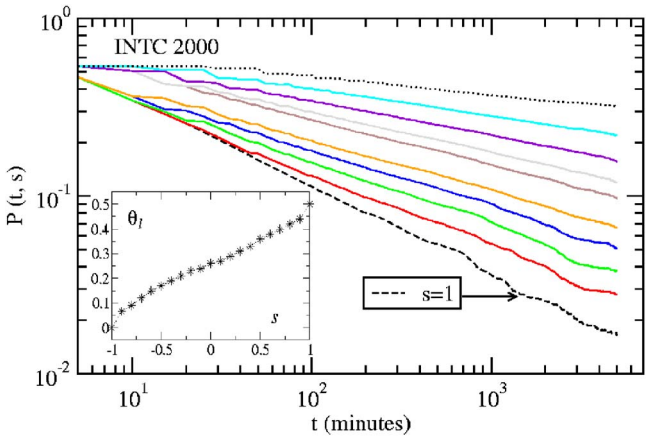

FIG. 13. (Color online) Log-log plot of the persistent large deviations probability $P(t,s)$ vs t for the INTC 2000 stock. The average sign parameter, s, is varied from 1 to −0.8 (bottom to top) with an increment of 0.2. The inset contains the persistent large deviations exponent θ_l vs s.

of 1 and 5 min, respectively. We have chosen different values of the measurement time t_m, which influences directly the value of the average price $\bar{x}=1/t_m\int_0^{t_m}dt'x(t')$. As shown in upper panel of Fig. 14 the survival probability measured over a longer t_m has a slower decrease with time than the one corresponding to a smaller t_m. We find that the empirical measurements of $S(t)$ show good scaling with $t/\delta t$ at short times ($t<300$ min), when the ration between the sampling time and measurement time is kept constant. This agrees with similar measurements done on experimental step fluctuations [24]. Since a very simple interpretation of the stock market fluctuations is based on the random walk model, we have numerically simulated the survival behavior of a random walker which allows us to understand qualitatively all the features of $S(t)$ found experimentally. Measurements of $S(t)$ for the random walk model were carried out for systems of size $L=100$. The measured average of the random walker variable at each site over the measurement time was used as the reference level in the calculation of the survival probability, and the results were averaged over 300 independent runs. In bottom panel of Fig. 14 we show that a perfect collapse of $S(t)$ vs $t/\delta t$ appears when the ratio $\delta t/t_m$ is constant. In addition, $S(t)$ corresponding to the data recorded with the same δt shows a slower decrease when the measurement time is larger, as in the empirical case. Therefore both sampling time and total measurement time have to be taken into consideration in order to interpret correctly the survival features of the financial systems. We want to point out that $S(t)$ does not show an exponential behavior over the investigated time range (see the inset of the bottom panel of Fig. 14), and possibly much larger t_m is needed to observe such a behavior at large time scales, as happens in the case of equilibrium surface step fluctuations [25].

IV. CONCLUSIONS

We conclude with some speculative thoughts on the possible development of "understanding" *in the sense of physics*

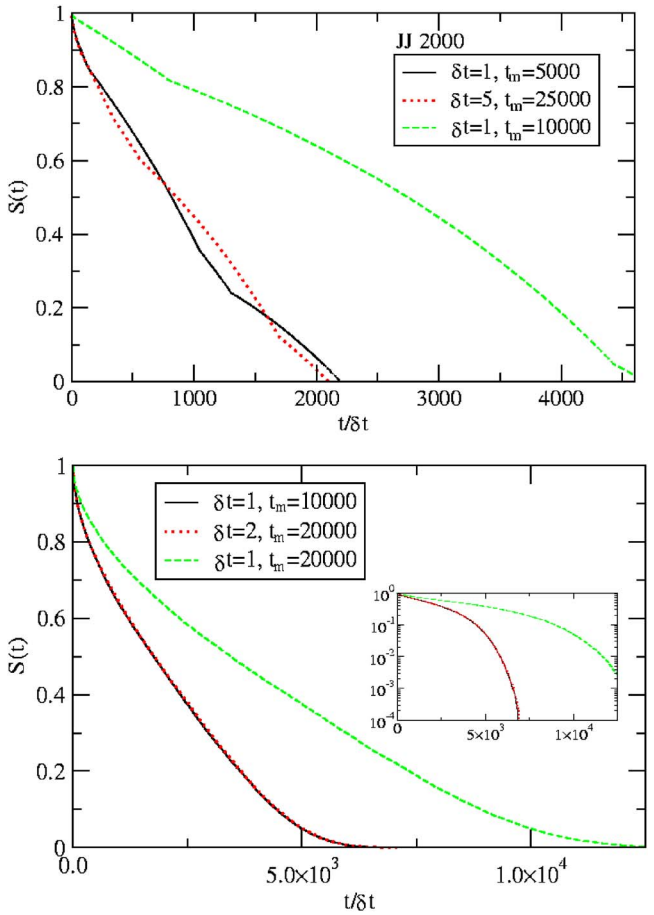

FIG. 14. (Color online) The average survival probability $S(t)$ vs $t/\delta t$ for the JJ 2000 stock (top) with different sampling times and different measurement times, as shown in the legend. The bottom plot contains the numerical simulation of $S(t)$ corresponding to a random walk with $L=100$. The scaling of $S(t)$ with $t/\delta t$ appears when $\delta t/t_m$ is constant. The inset shows the same curves on a logarithmic-linear scale.

with respect to stock price fluctuations. In physics—e.g., step fluctuations [15,16,23–25] or kinetic surface roughening [19]—one typically looks for minimal (in the renormalization group sense) dynamical (in general, nonlinear) stochastic continuum partial differential equations underlying the stochastic phenomena—e.g., the Edwards-Wilkinson equation [41], the Mullins-Herring equation [42], the Kardar-Parisi-Zhang equation [43], the Villain–Lai–Das Sarma equation [44,45], etc., hoping to derive the long-time coarse-grained asymptotic power-law behavior of the system arising from some simple minimal underlying dynamics (which is often based on symmetry and universality considerations). It is unclear whether such an approach based on continuum nonlinear stochastic equations is at all meaningful for the understanding of the dynamical evolution of complex economic phenomena such as stock price fluctuations. It may be possible to empirically construct dynamical equations which are *sufficient* to reproduce the exponents and the related statistical stochastic behavior described in this paper, but the *necessary* conditions for obtaining such equations are simply unknown (in fact, we do not know if such equations exist,

except in some "trivial" data fitting sense). It is therefore quite intriguing that stock price fluctuation data are amenable to stochastic analyses based on first-passage statistics and multifractality (as carried out in this article) with results not that dissimilar from physical processes such as step fluctuations or nonequilibrium growth.

Our work demonstrates that the persistence and the multifractal behaviors of stock prices (both individual and composite) are subtle (including extended self-similar properties not identified in the literature before). It will be of interest to investigate if the empirical behavior we report in this paper can be derived from the various multifractal stochastic models [10,11,13,14] for stock price fluctuations proposed in the literature. Such investigations, clearly beyond the scope of the current work, would, however, be quite difficult since both persistence and extended self-similarity are notoriously difficult concepts to derive theoretically, even when the underlying non-Markovian dynamics is known for a process. At this early stage of our understanding of econophysics, the fact that the stock price fluctuations seem to follow the persistence and the multifractal properties of well-studied surface fluctuation phenomena is by itself intriguing and interesting.

To summarize, in this study we have analyzed the multifractality, extended self-similarity, and first-passage properties of several financial stocks. While the second-order Hurst exponent and the persistence exponent characterizing the power-law decay of the average persistence probability are not able to explain the long-term correlations in the investigated price time series, higher-order correlation functions reveal much richer information about the complicated dynamics of such systems. We have shown that the persistence exponent θ is in agreement with $1-H_2$ and does not depend on the sampling time and measurement time. However, the survival probability has a nontrivial dependence on both δt and t_m, presenting scaling with $t/\delta t$ only when the ratio $\delta t/t_m$ is a constant. The numerical simulations of persistence, survival, and extended self-similarity features using discrete models of financial stocks remain an interesting open problem.

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